Supplementary material: Dynamic modeling of corporate credit ratings and defaults

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Contents

A.1 Figures 2

A.2 Tables 4

A.3 Simulation study 6

A.3.1 Simulation exercise 1: Estimation of model parameters . . . 6

A.3.2 Simulation exercise 2: Model selection . . . 7

A.4 Residual analysis on real data 7

A.4.1 Covariate dependent rater bias . . . 7

A.4.2 Rater time effect . . . 11

References 11

A.1 Figures

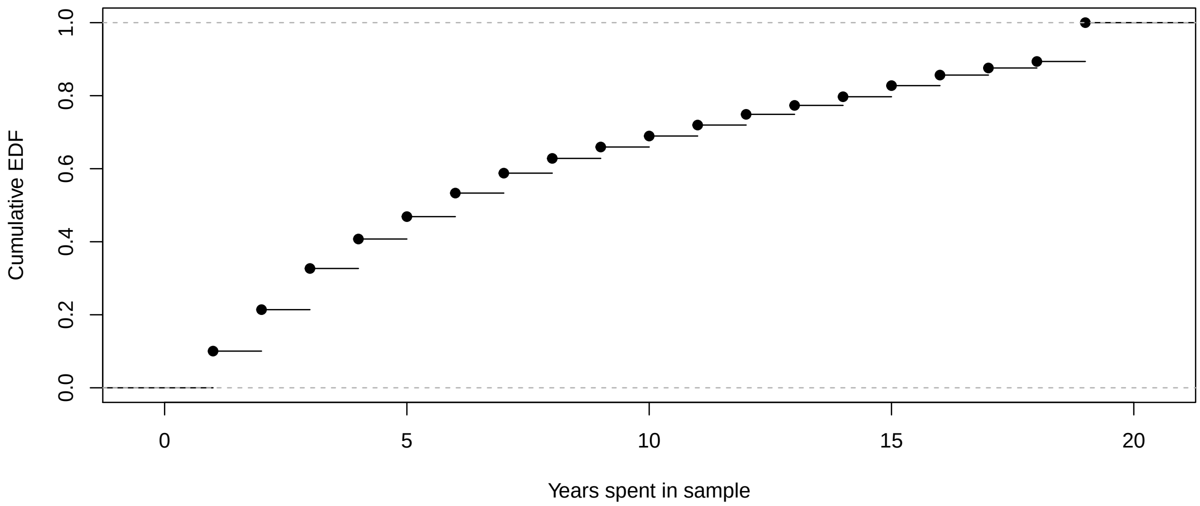

Figure A.1: This figure shows the empirical cumulative distribution function of the number of years a firm spends in the merged sample.

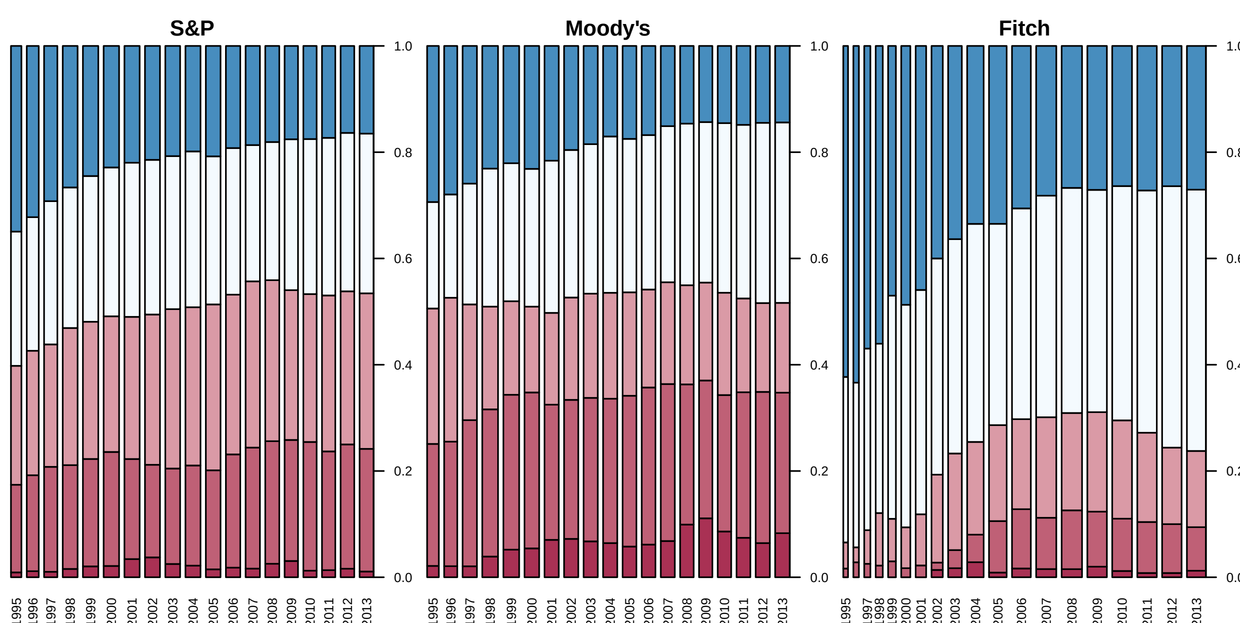

Figure A.2: This figure illustrates the rating distribution over the sample years for the three CRAs, where the red (blue) palette represents the speculative (investment) grades. The width of the bars is proportional to the number of rated firms.

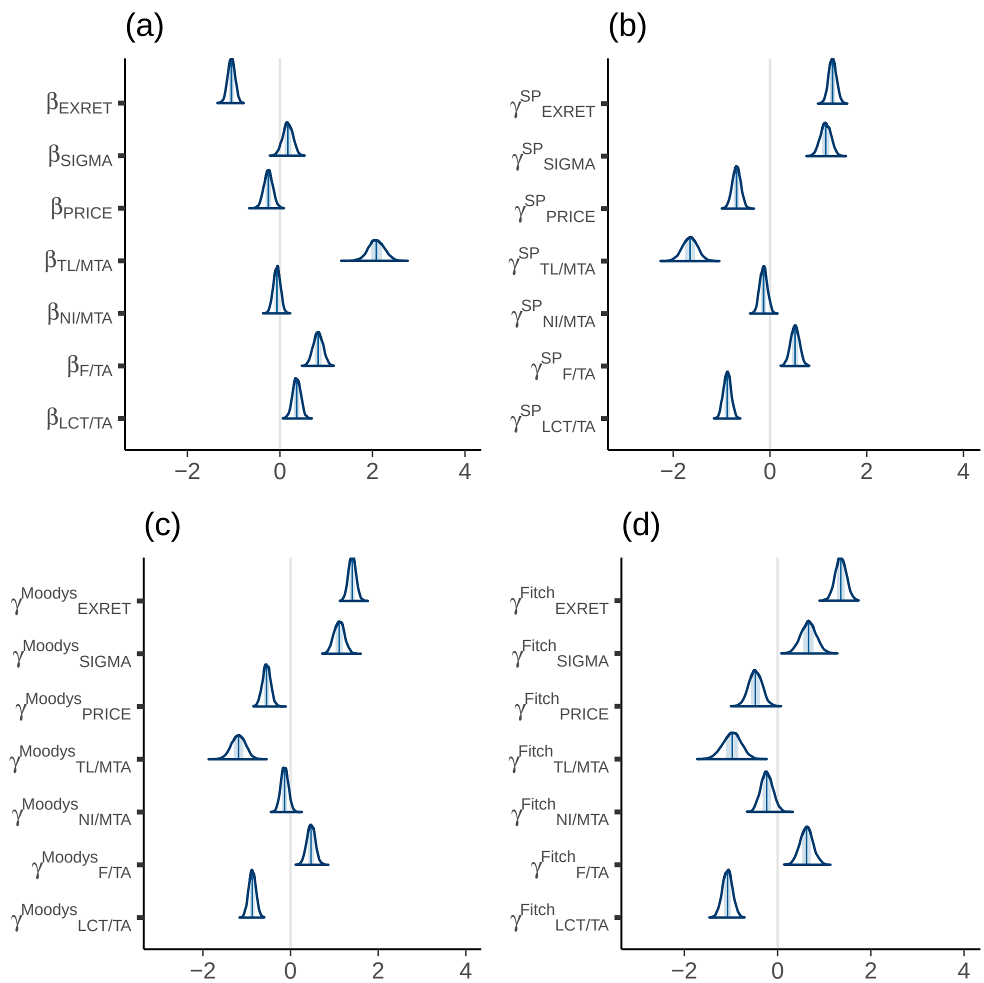

Figure A.3: This figure displays the posterior distributions with medians and 80% symmetric credible intervals of the regression coefficients in the latent creditworthiness equation (a) and of the regression coefficients corresponding to the rater bias of Standard and Poor's (b), Moody's (c) and Fitch (d).

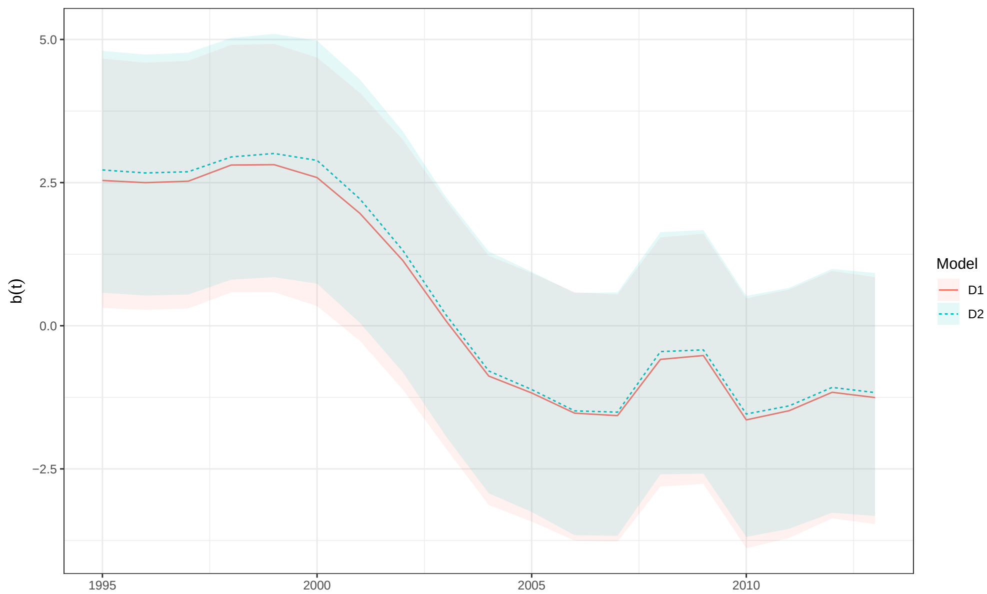

Figure A.4: This figure illustrates the posterior mean and 80% symmetric credible intervals for the market factor estimated for the dynamic models (D1) and (D2).

A.2 Tables

Table A.1: This table displays the summary statistics mean, lower quartile $Q_{0.25}$, median, upper quartile $Q_{0.75}$ and standard deviation (SD) of the covariates for both the entire sample and for the defaulted sample.

Statistic	Sample	LCT/TA	F/TA	NI/MTA	TL/MTA	PRICE	SIGMA	EXRET
Mean	Entire	0.2241	0.1975	0.0167	0.5167	2.4731	0.0249	−0.0584
	Defaulted	0.3465	0.4080	−0.1429	0.7708	1.8038	0.0589	−1.0972
$Q_{0.25}$	Entire	0.1340	0.1173	0.0039	0.3361	2.4594	0.0151	−0.2535
	Defaulted	0.1610	0.2607	−0.2307	0.6798	1.2342	0.0414	−1.5686
Median	Entire	0.2058	0.1785	0.0295	0.4988	2.7081	0.0209	−0.0140
	Defaulted	0.2552	0.3714	−0.1081	0.8205	1.8613	0.0569	−1.0937
$Q_{0.75}$	Entire	0.2853	0.2534	0.0460	0.6883	2.7081	0.0301	0.1965
	Defaulted	0.5117	0.5225	−0.0334	0.9005	2.6483	0.0775	−0.6002
SD	Entire	0.1239	0.1191	0.0787	0.2322	0.4639	0.0145	0.4770
	Defaulted	0.2445	0.1935	0.1400	0.1742	0.7635	0.0226	0.7730

Table A.2: This table displays the posterior mean (Mean), posterior standard deviation (SD) and the time series standard error (TS SE), as well as selected quantiles of the posterior distribution for various parameters in the dynamic model.

Parameter	Mean	SD	TS SE	$Q_{0.1}$	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	$Q_{0.9}$
β_0	−12.8838	1.0493	0.0148	−14.8030	−13.3789	−12.9513	−12.4977	−10.4639
$\beta_{\text{LCT/TA}}$	0.3624	0.0869	0.0012	0.1916	0.3034	0.3617	0.4216	0.5302
$\beta_{\text{F/TA}}$	0.8260	0.1046	0.0015	0.6228	0.7551	0.8256	0.8980	1.0291
$\beta_{\text{NI/MTA}}$	−0.0666	0.0765	0.0010	−0.2174	−0.1186	−0.0653	−0.0144	0.0802
$\beta_{\text{TL/MTA}}$	2.0869	0.1724	0.0024	1.7468	1.9716	2.0839	2.2037	2.4223
β_{PRICE}	−0.2505	0.0956	0.0014	−0.4334	−0.3137	−0.2503	−0.1870	−0.0629
β_{SIGMA}	0.1712	0.1076	0.0015	−0.0439	0.0989	0.1702	0.2464	0.3754
β_{EXRET}	−1.0500	0.0788	0.0011	−1.2038	−1.1035	−1.0490	−0.9959	−0.8988
$\gamma^{\text{S\&P}}_{\text{LCT/TA}}$	−0.8836	0.0785	0.0011	−1.0360	−0.9376	−0.8826	−0.8306	−0.7291
$\gamma^{\text{S\&P}}_{\text{F/TA}}$	0.5179	0.0913	0.0013	0.3378	0.4548	0.5183	0.5800	0.6949
$\gamma^{\text{S\&P}}_{\text{NI/MTA}}$	−0.1328	0.0781	0.0011	−0.2822	−0.1877	−0.1328	−0.0802	0.0220
$\gamma^{\text{S\&P}}_{\text{TL/MTA}}$	−1.6534	0.1564	0.0022	−1.9659	−1.7587	−1.6517	−1.5474	−1.3523
$\gamma^{\text{S\&P}}_{\text{PRICE}}$	−0.6916	0.0875	0.0012	−0.8650	−0.7509	−0.6908	−0.6327	−0.5228
$\gamma^{\text{S\&P}}_{\text{SIGMA}}$	1.1526	0.1103	0.0016	0.9405	1.0773	1.1517	1.2297	1.3651
$\gamma^{\text{S\&P}}_{\text{EXRET}}$	1.2938	0.0817	0.0012	1.1357	1.2382	1.2932	1.3474	1.4564
$\gamma^{\text{Moody's}}_{\text{LCT/TA}}$	−0.8727	0.0821	0.0012	−1.0349	−0.9294	−0.8735	−0.8173	−0.7102
$\gamma^{\text{Moody's}}_{\text{F/TA}}$	0.4672	0.0966	0.0015	0.2799	0.4021	0.4669	0.5323	0.6568
$\gamma^{\text{Moody's}}_{\text{NI/MTA}}$	−0.1361	0.0858	0.0012	−0.3031	−0.1952	−0.1368	−0.0781	0.0320
$\gamma^{\text{Moody's}}_{\text{TL/MTA}}$	−1.1884	0.1624	0.0023	−1.5115	−1.2969	−1.1870	−1.0767	−0.8747
$\gamma^{\text{Moody's}}_{\text{PRICE}}$	−0.5478	0.0925	0.0013	−0.7299	−0.6085	−0.5480	−0.4863	−0.3685
$\gamma^{\text{Moody's}}_{\text{SIGMA}}$	1.1086	0.1175	0.0017	0.8827	1.0263	1.1100	1.1889	1.3393
$\gamma^{\text{Moody's}}_{\text{EXRET}}$	1.4079	0.0866	0.0012	1.2406	1.3478	1.4082	1.4657	1.5785
$\gamma^{\text{Fitch}}_{\text{LCT/TA}}$	−1.0740	0.1081	0.0015	−1.2871	−1.1487	−1.0726	−1.0024	−0.8615
$\gamma^{\text{Fitch}}_{\text{F/TA}}$	0.6211	0.1391	0.0020	0.3488	0.5258	0.6214	0.7118	0.8995
$\gamma^{\text{Fitch}}_{\text{NI/MTA}}$	−0.2298	0.1308	0.0018	−0.4784	−0.3203	−0.2339	−0.1432	0.0337
$\gamma^{\text{Fitch}}_{\text{TL/MTA}}$	−0.9761	0.1973	0.0028	−1.3679	−1.1059	−0.9725	−0.8433	−0.5978
$\gamma^{\text{Fitch}}_{\text{PRICE}}$	−0.4739	0.1450	0.0021	−0.7529	−0.5723	−0.4770	−0.3753	−0.1825
$\gamma^{\text{Fitch}}_{\text{SIGMA}}$	0.6642	0.1674	0.0024	0.3395	0.5499	0.6639	0.7732	0.9959
$\gamma^{\text{Fitch}}_{\text{EXRET}}$	1.3564	0.1190	0.0017	1.1233	1.2752	1.3553	1.4383	1.5923
$\theta^{\text{S\&P}}_{\text{AAA/A}\|\text{BBB}}$	−16.0665	1.0057	0.0142	−17.9772	−16.7288	−16.0946	−15.4419	−14.0099
$\theta^{\text{S\&P}}_{\text{BBB}\|\text{BB}}$	−6.0050	0.9866	0.0140	−7.8909	−6.6464	−6.0428	−5.3809	−3.9914
$\theta^{\text{S\&P}}_{\text{BB}\|\text{B}}$	2.2271	0.9839	0.0139	0.3636	1.5864	2.1927	2.8441	4.2766
$\theta^{\text{S\&P}}_{\text{B}\|\text{CCC/C}}$	11.8137	1.0014	0.0145	9.9283	11.1497	11.7797	12.4485	13.8530
$\theta^{\text{Moody's}}_{\text{Aaa/A}\|\text{Baa}}$	−15.6523	1.1372	0.0161	−17.8437	−16.3980	−15.7002	−14.8936	−13.3536
$\theta^{\text{Moody's}}_{\text{Baa}\|\text{Ba}}$	−5.2233	1.1216	0.0159	−7.4193	−5.9494	−5.2798	−4.4805	−2.9605
$\theta^{\text{Moody's}}_{\text{Ba}\|\text{B}}$	1.0228	1.1208	0.0159	−1.1492	0.2830	0.9664	1.7531	3.2798
$\theta^{\text{Moody's}}_{\text{B}\|\text{Caa/Ca}}$	10.1974	1.1337	0.0160	7.9944	9.4626	10.1477	10.9344	12.4999
$\theta^{\text{Fitch}}_{\text{AAA/A}\|\text{BBB}}$	−11.6339	1.2528	0.0177	−14.1553	−12.4585	−11.6262	−10.7905	−9.2318
$\theta^{\text{Fitch}}_{\text{BBB}\|\text{BB}}$	−1.3525	1.2443	0.0176	−3.9116	−2.1513	−1.3577	−0.5136	1.0418
$\theta^{\text{Fitch}}_{\text{BB}\|\text{B}}$	6.0028	1.2774	0.0181	3.4262	5.1683	6.0086	6.8571	8.4314
$\theta^{\text{Fitch}}_{\text{B}\|\text{CCC/C}}$	14.1393	1.3523	0.0191	11.4413	13.2581	14.1302	15.0587	16.7452
q^2	60.1868	4.2536	0.0598	52.2788	57.2782	59.9816	62.8890	69.2222
ρ	0.9799	0.0019	0.0000	0.9759	0.9787	0.9800	0.9812	0.9834
ψ	1.8704	0.0352	0.0005	1.8023	1.8465	1.8702	1.8940	1.9394
ω	0.6938	0.2022	0.0082	0.3797	0.5536	0.6656	0.8006	1.1830
ϕ_b	0.7658	0.1609	0.0069	0.4173	0.6586	0.7777	0.8891	0.9952
$\lambda_{\text{Moody's}}$	1.1655	0.0322	0.0004	1.1037	1.1437	1.1653	1.1867	1.2293
λ_{Fitch}	1.2928	0.0638	0.0009	1.1691	1.2501	1.2906	1.3352	1.4246
ϕ_δ	0.9912	0.0066	0.0001	0.9748	0.9879	0.9926	0.9960	0.9992

A.3 Simulation study

We perform a simulation study to investigate how well the parameters of the five different models introduced in the manuscript can be recovered.

We simulate data sets from each of the five models where the latent creditworthiness is given by $S_i(t) = \beta_0 + \boldsymbol{\beta}^\top \boldsymbol{x}_i(t) + u_i(t)$, where we use different specifications for the random effect and for the rater bias:

Table A.3: Specification of random effect and rater bias for the models considered in the manuscript.

	Model *(PM)*	Model *(S1)*	Model *(S2)*	Model *(D1)*	Model *(D2)*
Random effect $u_i(t)$	$a_i - \omega b(t) + \epsilon_i(t)$, $a_i \sim N$, $b(t) \sim AR(1)$, $\epsilon_i(t) \sim AR(1)$	$\epsilon_i(t)$, $\epsilon_i(t) \sim N$	$\epsilon_i(t)$, $\epsilon_i(t) \sim N$	$a_i - \omega b(t) + \epsilon_i(t)$, $a_i \sim N$, $b(t) \sim AR(1)$, $\epsilon_i(t) \sim AR(1)$	$a_i - \omega b(t) + \epsilon_i(t)$, $a_i \sim N$, $b(t) \sim AR(1)$, $\epsilon_i(t) \sim AR(1)$
Rater bias $\eta_{ij}(t)$	$\boldsymbol{\gamma}_j^\top \boldsymbol{x}_i(t) + \lambda_j \delta_t$	0	$\boldsymbol{\gamma}_j^\top \boldsymbol{x}_i(t)$	0	$\boldsymbol{\gamma}_j^\top \boldsymbol{x}_i(t)$

For all data sets simulated data sets we use $I = 100$ firms, $T = 20$ time points and $P = 3$ covariates. For each firm-year we simulate a binary indicator D and ratings from three raters i.e., $J = \{R_1, R_2, R_3\}$. The three covariates X_1, X_2, X_3 are simulated from a standard normal distribution.

To incorporate the feature that our sample exhibits missing observations, we randomly remove 10% of the rating observations for R_1, 20% for R_2 and 30% for R_3.

The parameters used for simulating the data from the five models are the following:

- $\beta_0 = -2$, $\boldsymbol{\beta} = (0.2, 0.5, 1)^\top$,
- Each rater assigns ratings on a 4 point scale: $\boldsymbol{\theta}_1 = (-\infty, -3, -2, -1, \infty)^\top$, $\boldsymbol{\theta}_2 = (-\infty, -3, -2.5, -1.5, \infty)^\top$, $\boldsymbol{\theta}_3 = (-\infty, -4, -2, 0, \infty)^\top$
- $\boldsymbol{\gamma}_1 = (-0.1, -1, 0.5)^\top$, $\boldsymbol{\gamma}_2 = (-0.2, 0.5, -0.5)^\top$, $\boldsymbol{\gamma}_3 = (-0.8, 0, 1)^\top$
- $\psi = 0.5$, $\rho = 0.9$,
- $q^2 = 1$,
- $\phi_b = 0.7$, $\omega = 0.2$,
- $\lambda = (1, 0.2, 0.5)^\top$, $\phi_\delta = 0.5$.

We keep the hyperparameters of the priors constant over all model specifications. In each estimation step we use 5 chains of 2000 iterations, whereas the first 1000 are used as burn-in. Hence, inference is performed based on 5000 draws.

The codes for reproducing the simulation study can be found at the GitHub repository https://github.com/lauravana/dynamicmodelratingsdefaults.

A.3.1 Simulation exercise 1: Estimation of model parameters

For each model, we simulate 50 data sets from the model specification and estimate the parameters using **rstan**. For one data set, the root mean squared errors and the relative bias for each parameter κ is computed from the $m = 1, \dots, M$ posterior draws as:

$$RMSE(\kappa) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}(\kappa^{(m)} - \kappa^{\text{true}})^2}, \qquad RBIAS(\kappa) = \frac{1}{M}\sum_{m=1}^{M}\frac{(\kappa^{(m)} - \kappa^{\text{true}})}{\kappa^{\text{true}}}.$$

The distribution of the RMSEs is presented in Figure A.6 and the distribution of the relative bias in Figure A.5. We observe that parameters are well recovered in terms of bias. We see an overestimation in the persistence of the market factor ϕ_b, most likely due to the informative prior placed on this parameter, where high values of ϕ_b are preferred. The large RMSE of q^2 can be explained due to the fact that we only use $I = 100$ in the simulation so in this setting, inferring the value of the hyperparameter q^2 is associated with large uncertainty.

A.3.2 Simulation exercise 2: Model selection

We use for model comparison purposes the Bayesian LOO (leave-one-out) estimate of the expected log pointwise predictive (ELPD LOO) (Vehtari, Gelman, and Gabry 2017; Vehtari et al. 2020). For a single data set generated from each model, we estimate all five models in order to evaluate whether this information criterion is able to identify the correct model from which the data was simulated. Table A.4 contains the difference in ELPD LOO relative to the best estimated model, i.e., the model with highest ELPD LOO (a value of zero hence representing the best estimated model), together with an estimate for the standard error of the differences. The different rows correspond to data simulated from the true model. We observe that for each row, the true model is identified by this criterion. Even for the case where the data is generated from model ($D2$), while the best estimated model is model (PM), we can see that the difference to model ($D2$) is not significant and hence ($D2$) should be chosen, as it is more parsimonious.

Table A.4: This table contains the difference in the ELPD LOO relative to the best model among the five estimates ones.

	Estimated model: (S1)	Estimated model: (S2)	Estimated model: (D1)	Estimated model: (D2)	Estimated model: (PM)
True model : (S1)	**0.000 (0.000)**	-5.795 (2.416)	-14.500 (4.229)	-17.311 (4.285)	-23.514 (6.582)
True model : (S2)	-470.836 (29.108)	**0.000 (0.000)**	-479.325 (29.402)	-8.819 (3.842)	-22.383 (4.911)
True model : (D1)	-556.684 (29.912)	-559.300 (30.346)	**0.000 (0.000)**	-1.627 (4.012)	-4.823 (5.399)
True model : (D2)	-1046.650 (41.279)	-576.195 (29.772)	-439.735 (30.311)	**-0.445 (4.156)**	0.000 (0.000)
True model : (PM)	-1537.227 (45.219)	-1141.488 (38.073)	-780.563 (37.355)	-408.821 (26.783)	**0.000 (0.000)**

A.4 Residual analysis on real data

In order to investigate and motivate the specification of the rater bias, we perform an exploratory analysis on the residuals of various model specifications. We use surrogate residuals for ordinal regression problems (Liu and Zhang 2018) in our exploratory exercise, which are implemented in the R package **sure** (Greenwell, McCarthy, and Boehmke 2017).

In order to keep the computation time manageable, we estimate ordinal logit regression models in a frequentist setting and assume no random effects specification.

A.4.1 Covariate dependent rater bias

We first investigate whether there is a need for a covariate dependent rater bias by estimating the models:

$$\text{logit}\,\Pr(D = 1|X) = \beta_0 + \boldsymbol{\beta}^\top \boldsymbol{x}_i(t)$$

$$\text{logit}\,\Pr(R_j \leq r|X) = \theta_r - (\beta_0 + \boldsymbol{\beta}^\top \boldsymbol{x}_i(t)), \forall j \in \{\text{S\&P}, \text{Moody's}, \text{Fitch}\}.$$

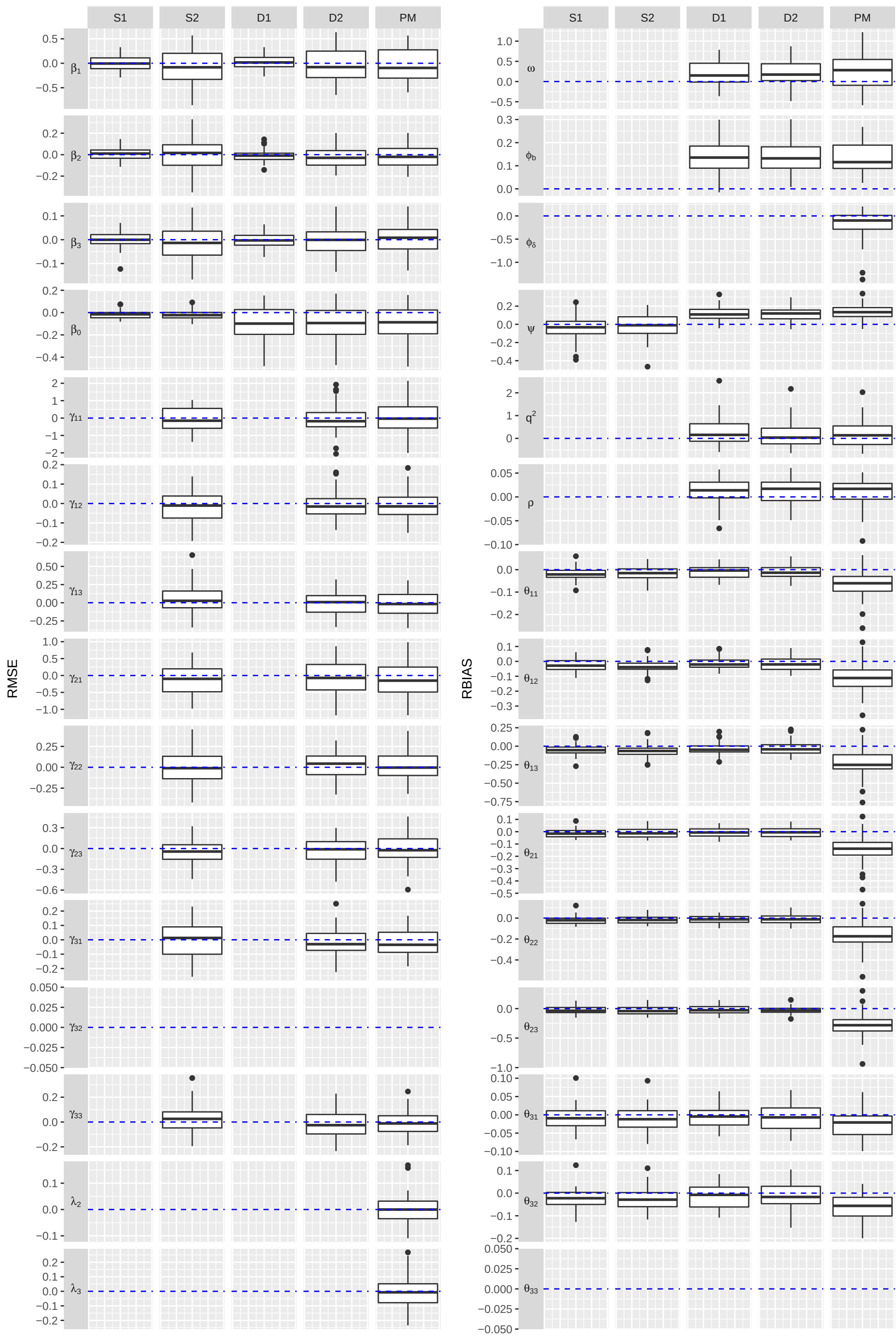

Figure A.5: Distribution of the RMSEs over the 50 simulated data sets from each of the five models (in case the true parameter is zero, no boxplot is shown).

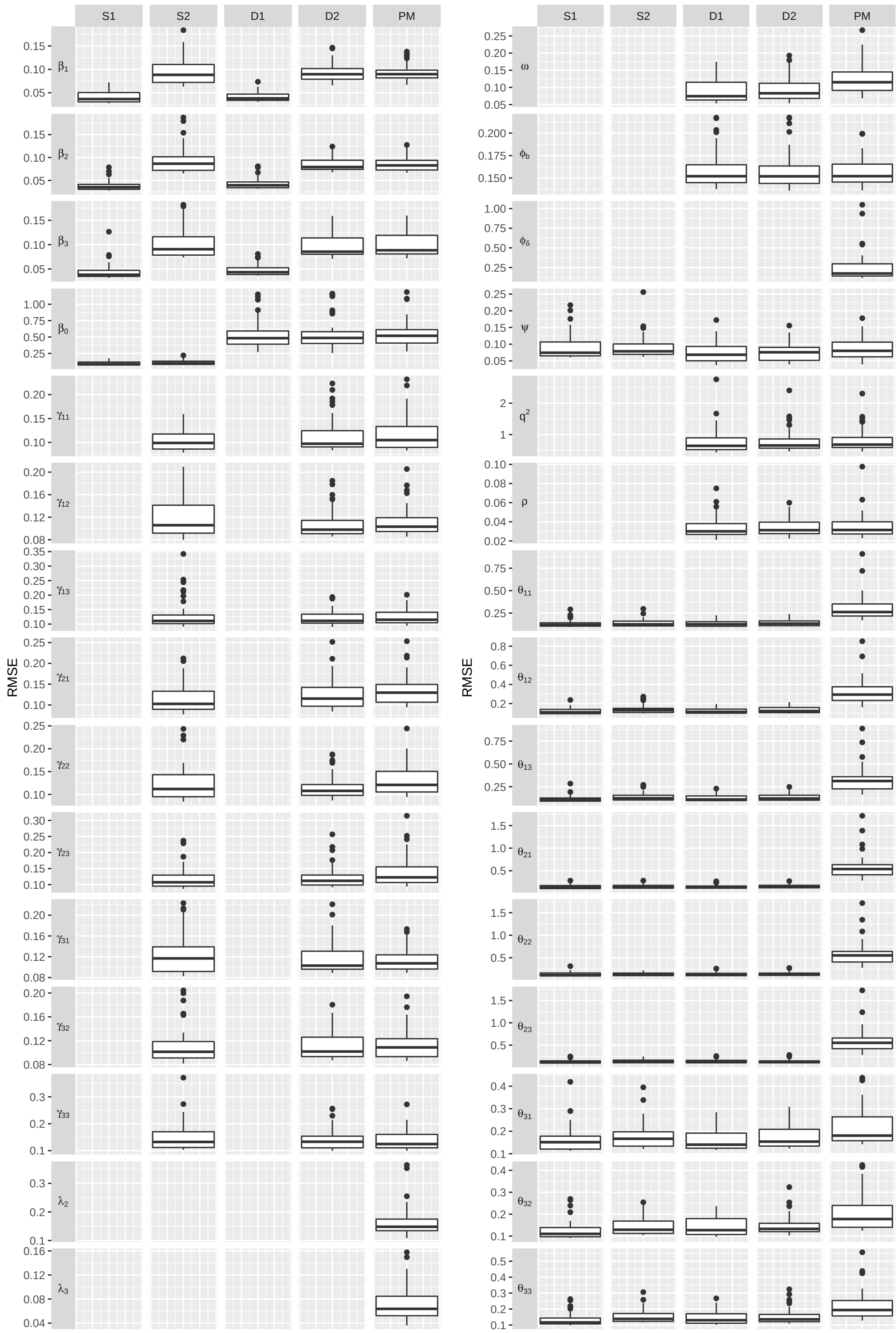

Figure A.6: Distribution of the relative bias over the 50 simulated data sets from each of the five models.

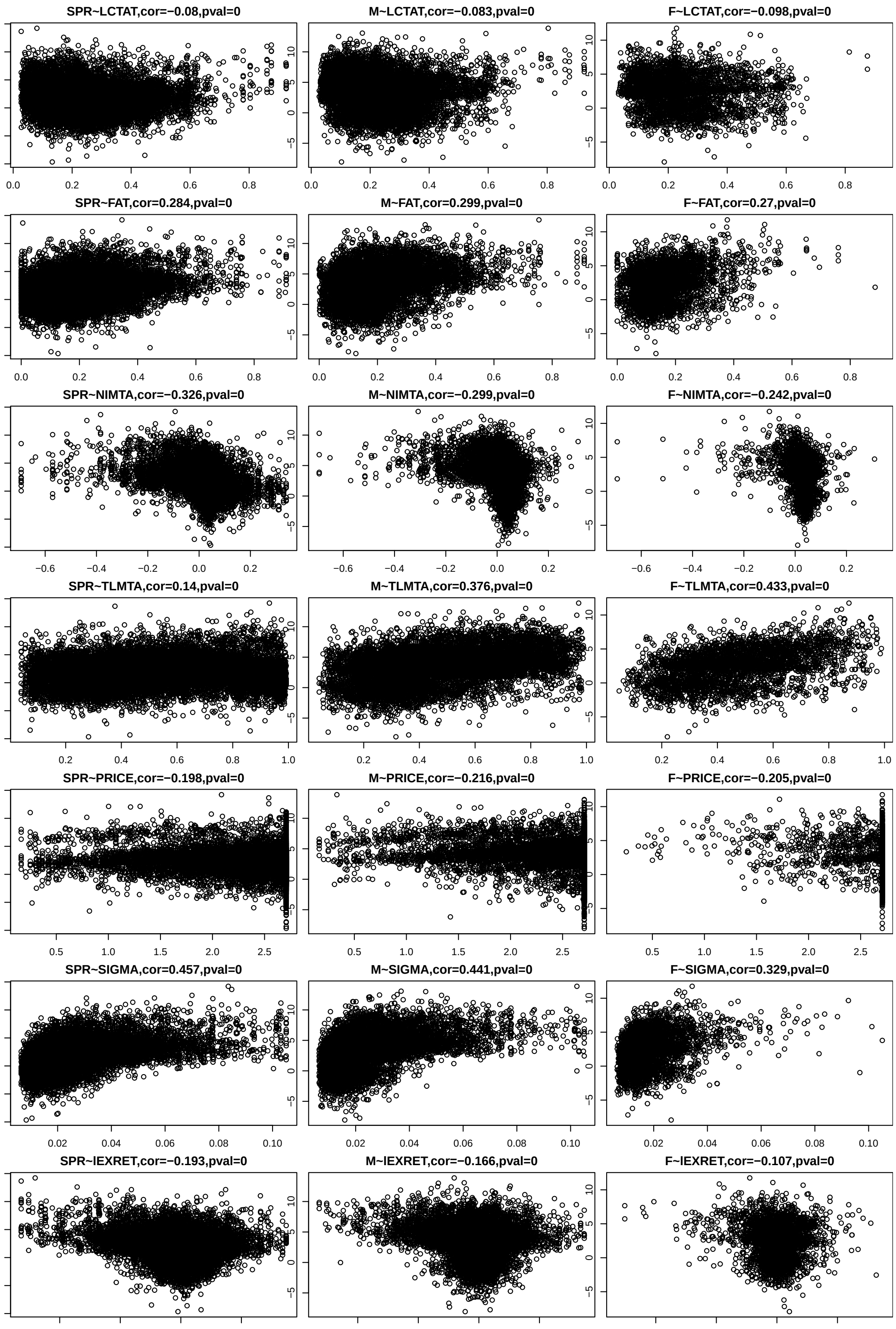

Figure A.7: Scatterplots of residuals and observed covariates, together with Pearsons correlation coefficient and the corresponding p-values. SPR stands for the residuals of the Standard and Poor's ratings, M for Moody's and F for Fitch.

where we assume a common set of $\boldsymbol{\beta}$ parameters (as in models *(S1)* and *(D1)*).

Figure A.7 shows the scatterplots of the residuals for each rater and of each covariate. We can observe that there are linear patterns between the residuals and the covariates unexplained by the models with a common set of $\boldsymbol{\beta}$.

A.4.2 Rater time effect

Next we estimate the models with a different set of regression coefficients for each response and investigate the time variation in the residuals. We aggregate the residuals for the four responses over time and present the results in Figure A.8. We observe a clear discrepancy among the time effect present in the default response and the time effect present in the ratings (which resembles a random walk, so we expect the coefficient of a corresponding AR(1) process to be close to 1). Moreover, the time trends in the residuals for the rating agencies resemble closely, which is why we specify a model with a common time rater factor.

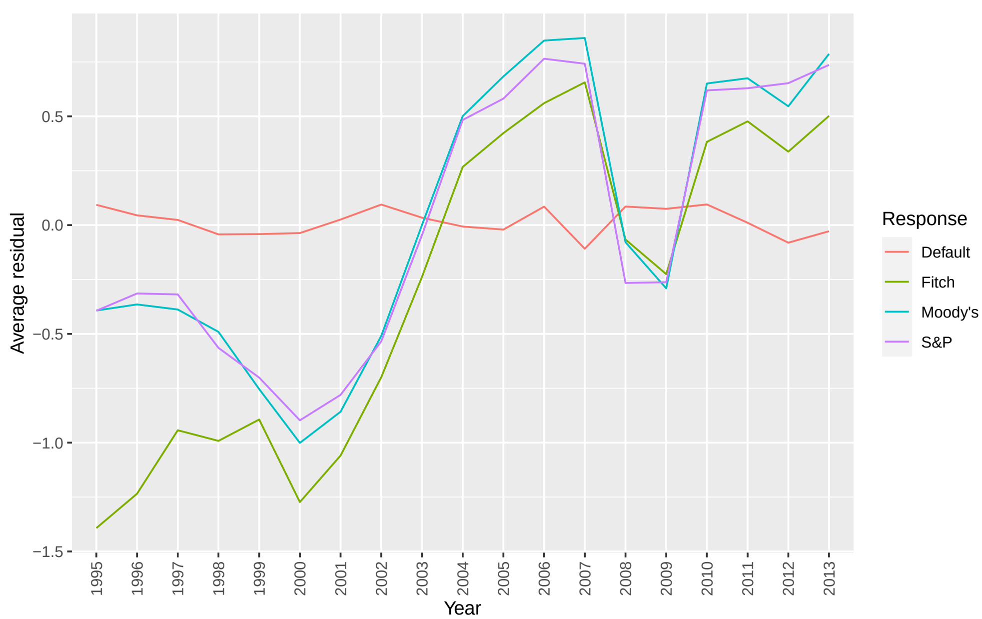

Figure A.8: Average residuals from ordinal logit regression models with separate regression coefficients.

References

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